Faculty of Engineering and Physical Sciences

Mathematics 1b: tutorial sheet 6

Exercise 1: Solve the following differential equations, giving the final answer in the form y=f(x) where possible

The differential equations in this exercise are first order homogeneous differential equations. What you have to do is to isolate expressions in x and y on either sign of the equal sign, then integrate on both sides without forgetting your integration constant and express y as a function of x. Finally if you are given some initial values such as what y is for, say x = 0, you have to use this to obtain the value of your integration constant. Having any (x,y) values makes your y function unique and provides you with the full solution of your differential equation.

(a)

$$x + \sin y \frac{dy}{dx} = 0$$

$$x = -\sin y \frac{dy}{dx}$$

$$xdx = -\sin ydy$$

$$\int xdx = \int -\sin ydy = \frac{x^2}{2} = \cos y + C$$

$$0 = \cos\frac{\pi}{4} + C \Rightarrow C = -\cos\frac{\pi}{4} = -\frac{\sqrt{2}}{2}$$

$$\frac{x^2}{2} = \cos y + C = \cos y - \frac{\sqrt{2}}{2} \Rightarrow \cos y = \frac{x^2}{2} + \frac{\sqrt{2}}{2} \Rightarrow y = \cos^{-1}(\frac{x^2}{2} + \frac{\sqrt{2}}{2})$$

It is worth noticing that it is easier for this particular exercise to introduce the value of y in the expression with "cosy" as it is much easier to obtain C at this stage.

(b)

In this problem, you have to remember how to do an integration by parts which is the way in which the integration in x should be solved.

Also you have to use the natural log function to obtain y. Please do remember that as Ln and exponential are reciprocal function, conjugating them means that you are applying the identity function to whatever is the power of your exponential:

$Ln(e^x) = x$

$Ln(e^{-y}) = Ln(1/e^y) = Ln1 - Lne^y = 0 - y = -y$

$$\frac{dy}{dx} = xe^{y-x} = xe^{y} \times e^{-x} = x\frac{e^{y}}{e^{x}} \Rightarrow \frac{dy}{e^{y}} = e^{-y}dy = \frac{x}{e^{x}}dx = xe^{-x}dx$$

$$\int e^{-y}dy = \int xe^{-x}dx$$

$$\int e^{-y}dy = -e^{-y}$$

$$\int xe^{-x}dx$$

$$u = x \Rightarrow du = dx$$

$$dv = e^{-x}dx \Rightarrow v = -e^{-x}$$

$$\int udv = [uv] - \int vdu$$

$$\int xe^{-x}dx = [-xe^{-x}] - \int -e^{-x}dx = -xe^{-x} + \int e^{-x}dx = -xe^{-x} - e^{-x} + C$$

$$-e^{-y} = -xe^{-x} - e^{-x} + C$$

$$e^{-y} = xe^{-x} + e^{-x} - C$$

$$Ln(e^{-y}) = Ln(xe^{-x} + e^{-x} - C) = -y \Rightarrow y = -Ln(xe^{-x} + e^{-x} - C)$$

(c)

$$\frac{dy}{dx} = \frac{xy - y}{y+1} = \frac{y(x-1)}{y+1} \Rightarrow (y+1)dy = y(x-1)dx \Rightarrow \frac{y+1}{y}dy = (x-1)dx$$

$$\int \frac{y+1}{y}dy = \int (x-1)dx = \int (1+\frac{1}{y})dy = y + Lny = \frac{x^2}{2} - x + C$$

$$y(2) = 1 \Rightarrow 1 + Ln1 = 1 = \frac{2^2}{2} - 2 + C \Rightarrow C = 1 + 2 - 2 = 1$$

$$y + Lny = \frac{x^2}{2} - x + 1$$

This particular solution cannot be rearranged in a more satisfactory manner; this is the most simple expression.

Exercise 2:
Slightly more complicated than the first exercise, those differential equations are still of the first order but are now inhomogeneous. There are two main ways to solve them and the one you have been taught involves the use of the integration factor, IF.
For any equation in the following form:

$$\frac{dy}{dx} + m(x)y = f(x)$$

The integration factor may be obtained as follows:

$$IF = e^{\int m(x)dx}$$

You then multiply the equation on both sides by IF and you should obtain a sum on the left hand side of the equation which is the result of a differentiation using the product rule. Your only problem then is to recognise which one. Then integrate without forgetting your integrating constant and finally express y as a function of x (or as a function of t as in question 2(d)). Just as a matter of interest, I will indicate another way of solving those equations that some of you may prefer. If you think that will confuse you even more, just do not have a look at it.

(a)

$$\frac{dy}{dx} + y = 5e^x$$

$$m(x) = 1 \Rightarrow IF = e^{\int dx} = e^x$$

$$e^x \frac{dy}{dx} + e^x y = 5e^x \times e^x = 5e^{2x}$$

$$\frac{d}{dx}(e^x y) = 5e^{2x} \Rightarrow e^x y = \frac{5}{2}e^{2x} + C \Rightarrow y = \frac{5e^{2x}}{2e^x} + \frac{C}{e^x} = \frac{5}{2}e^x + Ce^{-x}$$

(b)

$$\frac{dy}{dx} + \frac{1}{x}y = x^3 - 1$$

$$m(x) = \frac{1}{x} \Rightarrow IF = e^{\int \frac{1}{x}dx} = e^{Lnx} = x$$

$$x\frac{dy}{dx} + \frac{x}{x}y = x^4 - x = x\frac{dy}{dx} + y = \frac{d}{dx}(xy) \Rightarrow xy = \frac{x^5}{5} - \frac{x^2}{2} + C \Rightarrow y = \frac{x^5}{5x} - \frac{x^2}{2x} + \frac{C}{x}$$

$$y = \frac{x^4}{5} - \frac{x}{2} + \frac{C}{x}$$

$$y(1) = 2 \Rightarrow 2 = \frac{1}{5} - \frac{1}{2} + \frac{C}{1} \Rightarrow C = 2 - \frac{1}{5} + \frac{1}{2} = \frac{20 - 2 + 5}{10} = \frac{23}{10}$$

$$y = \frac{x^4}{5} - \frac{x}{2} + \frac{23}{10x}$$

(c)

$$\frac{dy}{dx}+\frac{2}{x+1}y=1$$

$$m(x)=\frac{2}{x+1}\Rightarrow IF=e^{\int\frac{2}{x+1}dx}=e^{2Ln(x+1)}=e^{Ln(x+1)^2}=(x+1)^2$$

$$(x+1)^2\frac{dy}{dx}+\frac{2(x+1)^2}{x+1}y=(x+1)^2=(x+1)^2\frac{dy}{dx}+2(x+1)y=(x+1)^2$$

$$\frac{d}{dx}((x+1)^2y)=(x+1)^2\Rightarrow(x+1)^2y=\frac{(x+1)^3}{3}+C\Rightarrow y=\frac{(x+1)^3}{3(x+1)^2}+\frac{C}{(x+1)^2}=\frac{(x+1)}{3}+\frac{C}{(x+1)^2}$$

(d)

$$t\frac{dy}{dt}+2y=e^t\Rightarrow\frac{dy}{dt}+\frac{2}{t}y=\frac{e^t}{t}$$

$$m(t)=\frac{2}{t}\Rightarrow e^{\int\frac{2}{t}dt}=e^{2Lxt}=e^{Lnt^2}=t^2$$

$$t^2\frac{dy}{dt}+\frac{2t^2}{t}y=t^2\frac{dy}{dt}+2ty=\frac{d}{dt}(t^2y)=te^t\Rightarrow t^2y=\int te^tdt=[te^t]-\int e^tdt=te^t-e^t+C$$

$$t^2y=te^t-e^t+C\Rightarrow y=\frac{te^t-e^t+C}{t^2}=\frac{e^t}{t}-\frac{e^t}{t^2}+\frac{C}{t^2}$$

As an example of another way of solving this type of equation, I will solve question 2(a) using the method, paradoxically called “variation of the constant”. This consists into solving the problem in two parts; first finding a general solution by making the equation homogenous (i.e. equalling the function on the right hand side of the sign equal to zero), then using this general solution and assuming that the constant in the expression now becomes a function of x (or t, when relevant). You then obtain a particular solution which you add up to the general solution to obtain the full solution.

$$\frac{dy}{dx} + y = 5e^x$$

(1)

$$\frac{dy}{dx} + y = 0 \Rightarrow \frac{dy}{dx} = -y \Rightarrow \frac{dy}{y} = -dx \Rightarrow Lny = -x + C \Rightarrow y = e^C e^{-x} = Ke^{-x}$$

(2)

$$\frac{dy}{dx} + y = 5e^x = \frac{dK}{dx}e^{-x} - K(x)e^{-x} + K(x)e^{-x} = \frac{dK}{dx}e^{-x} \Rightarrow \frac{dK}{dx}e^{-x} = 5e^x \Rightarrow \frac{dK}{dx} = \frac{5e^x}{e^{-x}} = 5e^{2x}$$

$$K(x) = \frac{5}{2}e^{2x} \Rightarrow K(x)e^{-x} = \frac{5}{2}e^x$$

(3)

$$y = \frac{5}{2}e^x + Ke^{-x}$$

Exercise 3:
In this exercise the difficulty is mostly two-fold: in the first instance, you have to recognise how to obtain your *IF* and then make sure you are using the initial value of the concentration c_0 correctly. Besides some of you also confused c(t) which is the concentration of glucose as a function of time with the integrating constant C. I will therefore use A instead for this particular exercise and write c as c(t) in all equations.

$$\frac{dc(t)}{dt} = \frac{G}{100V} - kc(t) \Rightarrow \frac{dc(t)}{dt} + kc(t) = \frac{G}{100V}$$

$$m(t) = e^{\int kdt} = e^{kt}$$

$$e^{kt}\frac{dc(t)}{dt} + ke^{kt}c(t) = \frac{G}{100V}e^{kt}$$

$$\frac{d}{dt}(e^{kt}c(t)) = \frac{G}{100V}e^{kt} \Rightarrow e^{kt}c(t) = \frac{G}{100kV}e^{kt} + A \Rightarrow c(t) = \frac{G}{100kV} \times \frac{e^{kt}}{e^{kt}} + \frac{A}{e^{kt}} = \frac{G}{100kV} + Ae^{-kt}$$

$$c(0) = c_0 \Rightarrow c_0 = \frac{G}{100kV} + A \Rightarrow A = c_0 - \frac{G}{100V}$$

$$c(t) = \frac{G}{100kV} + (c_0 - \frac{G}{100V})e^{-kt}$$

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