Faculty of Engineering and Physical Sciences

Mathematics 1b: tutorial sheet 9

Exercise 1:

Find the inverse of the matrix

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$

There are two methods to solve this.

1. Elimination
You first write the augmented matrix using the unit matrix I as follows:

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 2 & 1 & -1 & 0 & 1 & 0 \\ 1 & 2 & 5 & 0 & 0 & 1 \end{array}\right) \xrightarrow[R_3 \to R_3 - R_1]{R_2 \to R_2 - 2R_1} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 2 & 6 & -1 & 0 & 1 \end{array}\right) \xrightarrow{R_3 \to R_3 - 2R_2} \left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & -1 & 1 & 0 & 0 \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right) \xrightarrow{R_1 \to R_1 + \frac{1}{4}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 1 & -2 & 1 & 0 \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right) \xrightarrow{R_2 \to R_2 - \frac{1}{4}R_3} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right)$$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & 4 & 3 & -2 & 1 \end{array}\right) \xrightarrow{R_3 \to \frac{R_3}{4}} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & \frac{7}{4} & -\frac{1}{2} & \frac{1}{4} \\ 0 & 1 & 0 & -\frac{11}{4} & \frac{3}{2} & -\frac{1}{4} \\ 0 & 0 & 1 & \frac{3}{4} & -\frac{1}{2} & \frac{1}{4} \end{array}\right)$$

$$A^{-1} = \begin{pmatrix} \frac{7}{4} & \frac{-1}{2} & \frac{1}{4} \\ \frac{-11}{4} & \frac{3}{2} & \frac{-1}{4} \\ \frac{3}{4} & \frac{-1}{2} & \frac{1}{4} \end{pmatrix} = \frac{1}{4}\begin{pmatrix} 7 & -2 & 1 \\ -11 & 3 & -1 \\ 3 & -2 & 1 \end{pmatrix}$$

2. Transverse and comatrix
To calculate the inverse matrix this way, you have first to calculate the determinant corresponding to the matrix. This method has at least one advantage which is to tell you from the value of the determinant whether the matrix may be inverted or not and

straight away. If the determinant is equal to zero, this means that it is not possible to obtain an inverse matrix because it does not exist. It is also a systematic method that some of you may prefer because they do not always know how to get started to use the other method.

$$\Delta = \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{vmatrix} \overset{R_3 \to R_3 - R_1}{=} \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 2 & 6 \end{vmatrix} \overset{R_1 \to R_1 - \frac{1}{2}R_2}{=} \begin{vmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 0 & 2 & 6 \end{vmatrix} \overset{R_1 \to R_1 - \frac{1}{2}R_2}{=} \begin{vmatrix} 0 & -\frac{1}{2} & -\frac{1}{2} \\ 2 & 1 & -1 \\ 0 & 2 & 6 \end{vmatrix}$$

$$\Delta = -2 \times (-\frac{1}{2} \times 6 + \frac{1}{2} \times 2) = -2 \times (-3 + 1) = 4$$

The determinant of the above matrix is not equal to zero, therefore the inverse matrix exists. You now have to determine the comatrix of A; you then have to transpose this comatrix. This is done as follows:

$$A = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix}$$

To find out the values of numbers, you have to trace two lines, one horizontal and one vertical which pass through the number you have to replace to find the comatrix. For example, if you want to find the coefficient for the first row and the first column. Then each will be accompanied with a positive or negative sign in front of the result obtained, first row first column is + and then signs are alternated. Finally to find the transposed comatrix, one has to write the matrix obtained from a rotational symmetry with the central diagonal.

$$Com.(A) = \begin{pmatrix} +\begin{vmatrix} 1 & -1 \\ 2 & 5 \end{vmatrix} & -\begin{vmatrix} 2 & -1 \\ 1 & 5 \end{vmatrix} & +\begin{vmatrix} 2 & 1 \\ 1 & 2 \end{vmatrix} \\ -\begin{vmatrix} 0 & -1 \\ 2 & 5 \end{vmatrix} & +\begin{vmatrix} 1 & -1 \\ 1 & 5 \end{vmatrix} & -\begin{vmatrix} 1 & 0 \\ 1 & 2 \end{vmatrix} \\ +\begin{vmatrix} 0 & -1 \\ 1 & -1 \end{vmatrix} & -\begin{vmatrix} 1 & -1 \\ 2 & -1 \end{vmatrix} & +\begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} \end{pmatrix} = \begin{pmatrix} 7 & -11 & 3 \\ -2 & 6 & -2 \\ 1 & -1 & 1 \end{pmatrix}$$

$$Tr(Com(A)) = \begin{pmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{pmatrix}$$

$$A^{-1} = \frac{1}{\Delta} Tr(Com(A)) = \begin{pmatrix} \frac{7}{4} & \frac{-2}{4} & \frac{1}{4} \\ \frac{-11}{4} & \frac{6}{4} & \frac{-1}{4} \\ \frac{3}{4} & \frac{-2}{4} & \frac{1}{4} \end{pmatrix} = \begin{pmatrix} \frac{7}{4} & \frac{-1}{2} & \frac{1}{4} \\ \frac{-11}{4} & \frac{3}{2} & \frac{-1}{4} \\ \frac{3}{4} & \frac{-1}{2} & \frac{1}{4} \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{pmatrix}$$

The inverse matrix can be used to obtain the solution to the system indicated in this exercise as the matrix A corresponds exactly to this system.

$$\begin{cases} x_1 - x_3 = 2 \\ 2x_1 + x_2 - x_1 = 4 \\ x_1 + 2x_2 + 5x_3 = 14 \end{cases}$$

or

$$\begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & -1 \\ 1 & 2 & 5 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} = A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}$$

Therefore

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix}$$

and

$$A^{-1} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix}$$

$$A^{-1} A = I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

So

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 7 & -2 & 1 \\ -11 & 6 & -1 \\ 3 & -2 & 1 \end{pmatrix} \begin{pmatrix} 2 \\ 4 \\ 14 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 14 - 8 + 14 \\ -22 + 24 - 14 \\ 6 - 8 + 14 \end{pmatrix} = \frac{1}{4} \begin{pmatrix} 20 \\ -12 \\ 12 \end{pmatrix}$$

Therefore

$$\begin{cases} x_1 = 5 \\ x_2 = -3 \\ x_3 = 3 \end{cases}$$

Exercise 2:

For calculation method see tutorial 8.

$$A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

$$A^2 = \begin{pmatrix} 1\times1-1\times3+2\times0 & 1\times(-1)-1\times2+2\times1 & 1\times2-1\times4+2\times(-2) \\ 3\times1+2\times3+4\times0 & 3\times(-1)+2\times2+4\times1 & 3\times2+2\times4+4\times(-2) \\ 0\times1+1\times3-2\times0 & 0\times(-1)+1\times2-2\times1 & 0\times2+1\times4-2\times(-2) \end{pmatrix}$$

$$A^2 = \begin{pmatrix} -2 & -1 & -6 \\ 9 & 5 & 6 \\ 3 & 0 & 8 \end{pmatrix}$$

and

$$A^3 = A^2 \times A = A \times A^2 = \begin{pmatrix} -2 & -1 & -6 \\ 9 & 5 & 6 \\ 3 & 0 & 8 \end{pmatrix} \times \begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -2\times1-1\times3-6\times0 & -2\times(-1)-1\times2-6\times1 & -2\times2-1\times4-6\times(-2) \\ 9\times1+5\times4+6\times0 & 9\times(-1)+5\times2+6\times1 & 9\times2+5\times4+6\times(-2) \\ 3\times1+0\times3+8\times0 & 3\times(-1)+0\times2+8\times1 & 3\times2+0\times4+8\times(-2) \end{pmatrix}$$

$$A^3 = \begin{pmatrix} -5 & -6 & 4 \\ 24 & 7 & 26 \\ 3 & 5 & -10 \end{pmatrix}$$

Then

$$A^3 - A^2 - 5A + 8I = \begin{pmatrix} -5 & -6 & 4 \\ 24 & 7 & 26 \\ 3 & 5 & -10 \end{pmatrix} - \begin{pmatrix} -2 & -1 & -6 \\ 9 & 5 & 6 \\ 3 & 0 & 8 \end{pmatrix} - 5\begin{pmatrix} 1 & -1 & 2 \\ 3 & 2 & 4 \\ 0 & 1 & -2 \end{pmatrix} + 8\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} =$$

$$A^3 - A^2 - 5A + 8I = \begin{pmatrix} -5+2-5+8 & -6+1+5 & 4+6-10 \\ 24-9-15 & 7-5-10+8 & 26-6-20 \\ 3-3 & 5-5 & -10-8+10+8 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = 0$$

$$A^3 - A^2 - 5A + 8I = 0$$

Please note that in this particular expression “0” is not the number “zero” but a zero matrix, i.e. a three rows, three columns matrix with only zero as numbers.
We can use the above expression further to find the value of the inverse matrix of A.
If we multiply this expression by A^{-1} we obtain the following:

$$A^3 - A^2 - 5A + 8I = 0$$
$$A^{-1}A^3 - A^{-1}A^2 - 5A^{-1}A + 8A^{-1}I = 0$$
$$A^{-1}A = I$$
$$A^{-1}A^3 - A^{-1}A^2 - 5A^{-1}A + 8A^{-1}I = A^{-1}AA^2 - A^{-1}AA - 5A^{-1}A + 8A^{-1}I = A^2 - A - 5I + 8A^{-1} = 0$$

You also have to remember that any matrix multiplied by the identity matrix is the matrix itself. We can now use the expression we deduced to obtain A^{-1}, all you need to do is to consider the expression like any common algebraic expression and deduce A^{-1}:

$$A^2 - A - 5I + 8A^{-1} = 0$$
$$A^{-1} = \frac{1}{8}\left(5I + A - A^2\right) = \frac{1}{8}\begin{pmatrix} 5+1-(-2) & -1+1 & 2+6 \\ 3-9 & 5+2-5 & 4-6 \\ -3 & 1 & 5-2-8 \end{pmatrix} = \frac{1}{8}\begin{pmatrix} 8 & 0 & 8 \\ -6 & 2 & -2 \\ -3 & 1 & -5 \end{pmatrix}$$

If you have any doubt, you can always try and multiply A by its inverse A^{-1} and see if you obtain the matrix identity I.

Exercise 3:

To solve the proposed system, you can use the augmented matrix:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 2x_1 + x_2 = -1 \\ -2x_1 + 3x_2 + 8x_3 = 13 \end{cases}$$

$$\left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & -1 \\ -2 & 3 & 8 & 13 \end{array}\right) \xrightarrow{R3 \to R3+R2} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 2 & 1 & 0 & -1 \\ 0 & 4 & 8 & 12 \end{array}\right) \xrightarrow{R2 \to R2-2R1} \left(\begin{array}{ccc|c} 1 & 2 & 3 & 4 \\ 0 & -3 & -6 & -9 \\ 0 & 4 & 8 & 12 \end{array}\right)$$

It is now obvious that row 2 and 3 are equivalent and this means that there is not a unique solution but an infinity of solutions. This is because the matrix is singular. Therefore we shall impose $x_3 = \alpha$ and deduce the other unknowns:

$$\begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ 4x_2 + 8x_3 = 12 \\ x_3 = \alpha \end{cases} \Rightarrow \begin{cases} x_1 + 2x_2 + 3x_3 = 4 \\ x_2 + 2x_3 = 3 \\ x_3 = \alpha \end{cases} \Rightarrow \begin{cases} x_1 = 4 - 2(3-2\alpha) - 3\alpha = -2 + \alpha \\ x_2 = 3 - 2\alpha \\ x_3 = \alpha \end{cases}$$

Exercise 4:

(i)

$$\Delta = \begin{vmatrix} 2 & -1 & 7 \\ 4 & 6 & -2 \\ 5 & 8 & -3 \end{vmatrix} \overset{R_2 \to R_2 - 2R_1}{=} \begin{vmatrix} 2 & -1 & 7 \\ 0 & 8 & -16 \\ 5 & 8 & -3 \end{vmatrix} = 2(-3\times 8 + 16\times 8) + 5(16 - 8\times 7) = 208 - 200 = 8$$

(ii)

$$\Delta = \begin{vmatrix} -5 & 2 & 4 \\ 1 & -3 & 4 \\ 0 & 1 & 3 \end{vmatrix} \overset{R_1 \to R_1 + 5R_2}{=} \begin{vmatrix} 0 & -13 & 24 \\ 1 & -3 & 4 \\ 0 & 1 & 3 \end{vmatrix} = -1(-13\times 3 - 24) = 63$$

(iii)

$$\Delta = \begin{vmatrix} 2 & 5 & -3 \\ 3 & -2 & 1 \\ 7 & 8 & -5 \end{vmatrix} \overset{R_3 \to R_3 - 2R_1 - R_3}{=} \begin{vmatrix} 2 & 5 & -3 \\ 3 & -2 & 1 \\ 0 & 0 & 0 \end{vmatrix} = 0$$

As the manipulation of rows provides a row with only zeros, the determinant is equal to zero and the matrix is therefore singular.

Exercise 5:

$$\Delta = \begin{vmatrix} 2x+y+z & x & x^2 \\ x+2y+z & y & y^2 \\ x+y+2z & z & z^2 \end{vmatrix} \overset{C_1 \to C_1 - C_2}{=} \begin{vmatrix} x+y+z & x & x^2 \\ x+y+z & y & y^2 \\ x+y+z & z & z^2 \end{vmatrix} = (x+y+z)\begin{vmatrix} 1 & x & x^2 \\ 1 & y & y^2 \\ 1 & z & z^2 \end{vmatrix}$$

$$\Delta \overset{\substack{R_2 \to R_2 - R_1 \\ R_3 \to R_3 - R_1}}{=} (x+y+z)\begin{vmatrix} 1 & x & x^2 \\ 0 & y-x & y^2-x^2 \\ 0 & z-x & z^2-x^2 \end{vmatrix} = (x+y+z)\left[(y-x)(z^2-x^2) - (z-x)(y^2-x^2)\right]$$

$$\Delta = (x+y+z)\left[(y-x)(z-x)(z+x) - (z-x)(y-x)(y+x)\right]$$

$$\Delta = (x+y+z)(y-x)(z-x)\left[(z+x) - (y+x)\right] = (x+y+z)(y-x)(z-x)(z-y)$$

$$\Delta = (x+y+z)(x-y)(y-z)(z-x)$$

Firstly you should try and get as many zeros within the determinant, then remember identities. Also do not forget when a common factor is present in either row or column of a determinant it can be factorised (as opposed to matrices where a common factor has to be present for all numbers if you want to factorise anything).