Calculus (Spring) Sheet 3 solutions

1. Use the Lagrange multiplier method. $\nabla f = \lambda \nabla g$ gives $(3, -2, 1) = \lambda(2x, 2y, 2z)$ so that

$$x = \frac{3}{2\lambda}, \quad y = -\frac{1}{\lambda}, \quad z = \frac{1}{2\lambda}.$$

Putting these into the constraint gives

$$\frac{9}{4\lambda^2} + \frac{1}{\lambda^2} + \frac{1}{4\lambda^2} = 14$$

so that $\lambda = \frac{1}{2}$ or $-\frac{1}{2}$ giving $(x, y, z) = (3, -2, 1)$ or $(-3, 2, -1)$. The first of these gives the maximum value (just try both of them), giving the maximum value to be $f(3, -2, 1) = 14$.

2. Let the vertices be at (x, y, z), $(-x, y, z)$, $(x, -y, z)$, $(-x, -y, z)$, $(x, y, -z)$, $(-x, y, -z)$, $(x, -y, -z)$ and $(-x, -y, -z)$. Then the volume V is given by $V = 8xyz$. Applying the method of Lagrange multipliers, we must solve $\nabla V = \lambda \nabla g$ where $g(x, y, z) = 4x^2 + 9y^2 + 36z^2 - 36$, giving

$$(8yz, 8xz, 8xy) = \lambda(8x, 18y, 72z)$$

so that $8yz = 8\lambda x$, $8xz = 18\lambda y$ and $8xy = 72\lambda z$. Multiplying the first of these by x, the second by y and the third by z gives $8x^2 = 18y^2 = 72z^2$ so that $z = x/3$ and $y = 2x/3$. Putting these into the constraint $4x^2 + 9y^2 + 36z^2 = 36$, gives

$$4x^2 + 9\left(\frac{2x}{3}\right)^2 + 36\left(\frac{x}{3}\right)^2 = 36$$

so that $x = \sqrt{3}$, $y = 2\sqrt{3}/3$ and $z = \sqrt{3}/3$. Putting these values into the volume formula $V = 8xyz$ gives the volume to be $16\sqrt{3}/3$.

3. Let $f(x, y, z) = 4x^3y^2 + 2y - z$ so the surface is given by $f(x, y, z) = 0$. The vector ∇f is normal to the surface at every point. So a normal to $f(x, y, z) = 0$ is $\nabla f = (12x^2y^2, 8x^3y + 2, -1)$. At the point $(1, -2, 12)$ this gives $\nabla f = (48, -14, -1)$. So the tangent plane will have equation $(\mathbf{r} - \mathbf{a}) \cdot \mathbf{n} = 0$ with $\mathbf{a} = (1, -2, 12)$ and $\mathbf{n} = (48, -14, -1)$. This gives

$$((x, y, z) - (1, -2, 12)) \cdot (48, -14, -1) = 0$$

which becomes $48(x - 1) - 14(y + 2) - (z - 12) = 0$ or $48x - 14y - z = 64$.

5.

$$\begin{aligned}
\int_0^1 \int_{y^2}^{y} \sqrt{xy}\, dx\, dy &= \int_0^1 \int_{y^2}^{y} x^{1/2} y^{1/2}\, dx\, dy \\
&= \int_0^1 \left[\frac{x^{3/2}}{3/2} y^{1/2} \right]_{x=y^2}^{x=y} dy \\
&= \int_0^1 \left(\frac{2}{3} y^2 - \frac{2}{3} y^{7/2} \right) dy = \frac{2}{27}
\end{aligned}$$

$$\begin{aligned}\int_1^\infty \int_{e^{-x}}^1 \frac{1}{x^3 y}\, dy\, dx &= \int_1^\infty \left[\frac{1}{x^3}\ln y\right]_{y=e^{-x}}^{y=1} dx \\ &= \int_1^\infty \frac{1}{x^2}\, dx = 1\end{aligned}$$

6. For part (i) it is slightly easier to take $dA = dy\, dx$ than $dx\, dy$.

$$\begin{aligned}\iint_D (x+y)\, dA &= \int_0^1 \int_{x^4}^{x^3} (x+y)\, dy\, dx \\ &= \int_0^1 \left[xy + \frac{y^2}{2}\right]_{y=x^4}^{y=x^3} dx \\ &= \int_0^1 \left(x^4 + \frac{x^6}{2} - x^5 - \frac{x^8}{2}\right) dx = \frac{31}{630}\end{aligned}$$

Part (ii) can only be done taking $dA = dy\, dx$:

$$\begin{aligned}\iint_D e^{x^2}\, dA &= \int_0^2 \int_0^{x/2} e^{x^2}\, dy\, dx \\ &= \int_0^2 \frac{x}{2} e^{x^2}\, dx = \left[\frac{1}{4}e^{x^2}\right]_0^2 = \frac{1}{4}(e^4 - 1)\end{aligned}$$

If you don't like the idea of integrating $\frac{x}{2}e^{x^2}$ by inspection, do it by substituting $t = x^2$.

7. This is similar to an example done in lectures, although here we will make a substitution which makes it easier. Letting D be the region in the (x, y) plane given by $0 \le y \le b(1 - x/a)$ for $0 \le x \le a$, the volume is

$$\iint_D c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\, dy$$

Let $x = au$, $y = bv$ and transform into the new variables (u, v). The Jacobian of the transformation is ab, and in terms of the (u, v) variables the integral becomes

$$\begin{aligned}\iint_D c\left(1 - \frac{x}{a} - \frac{y}{b}\right) dx\, dy &= abc \int_0^1 \int_0^{1-v} (1 - u - v)\, du\, dv \\ &= abc \int_0^1 \left(1 - v - \frac{(1-v)^2}{2} - v(1-v)\right) dv \\ &= abc \int_0^1 \frac{(1-v)^2}{2}\, dv = \frac{abc}{6}\end{aligned}$$