Numbers, functions and analysis: further exercises

This sheet must not be used in isolation. You need to revise the notes and exercise sheets as well. Some exam questions may ask you to state and prove theorems, or to give formal definitions.

1. Prove by induction that $2^{2n} - 1$ is divisible by 3 for every positive integer n.

2.

(i) Find the highest common factor of 2322 and 654.

(ii) Show that $\mathrm{hcf}(234, 180) = 18$, and find integers x and y such that $18 = 234x + 180y$.

3. Prove that, for any integer m,

$$m^2 \equiv 0 \text{ or } 1 \text{ or } 4 \text{ or } 7 \pmod 9.$$

Prove that no number whose digits add up to 15 can be a perfect square. [Hint: remember that, modulo 9, any number is congruent to the sum of its digits].

4. Show that, for any integers a and b, $a^2 + b^2 \equiv 0$ or 1 or 2 $\pmod 4$. Hence show that 4 926 834 923 cannot be written as a sum of two squares.

5. Show that the sequence $a_1 = 1$, $a_{n+1} = \sqrt{6 + a_n}$ is a monotonically increasing sequence satisfying $a_n \leq 3$ for all n. Deduce that (a_n) converges and find the limit of this sequence.

6. Prove, for $n = 1, 2, 3, \ldots$, that $2^n/n! \leq 4/n$. Hence evaluate $\lim_{n\to\infty} 2^n/n!$.

7. Show that the equation $2x^3 - 4x^2 + 5x - 4 = 0$ has a root between 1 and 2. Which of these numbers is the root closer to?

8. State Rolle's theorem. Let $f(x)$ be continuous and differentiable. If the equation $f(x) = 0$ has four distinct real roots show that there must be a value of x for which $f'''(x) = 0$.

9. Let f, g be continuous on $[a, b]$ and differentiable on (a, b) and assume $g(a) \neq g(b)$. By applying Rolle's theorem to the function

$$H(x) = (f(b) - f(a))g(x) - (g(b) - g(a))f(x),$$

prove that there exists $c \in (a, b)$ such that

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}.$$

10. A function f is twice differentiable and satisfies $f''(x) \geq 0$. If $a < b$ and $0 \leq \lambda \leq 1$ show that

$$f(\lambda a + (1 - \lambda)b) \leq \lambda f(a) + (1 - \lambda)f(b).$$

[Apply the mean value theorem twice, once on the interval $[a, \lambda a + (1 - \lambda)b]$ and once on $[\lambda a + (1 - \lambda)b,\, b]$].

Continued overleaf

11. Determine whether the following series converge or diverge

(i) $\sum_{n=1}^{\infty} \frac{n}{1+n^2}$;

(ii) $\sum_{n=1}^{\infty} n\, e^{-n^2}$;

(iii) $\sum_{n=1}^{\infty} \frac{1}{\sqrt{1+n^2}}$.

Answers:

2. (i) 6, (ii) $x = -3$, $y = 4$.

5. 3

6. 0

7. 1

11. (i) diverges, (ii) converges, (iii) diverges