- insects: metamorphic moult triggered by <u>size</u> (not age) of a larva.

- developmental time affected by intra-specific competition.

$i(x,t)$: density of larvae of size $\overset{\text{length}}{x}$ at time t

$$I(t) = \int_0^l i(x,t)\,dx \qquad l = \text{threshold length.}$$

larvae grow such that $\dfrac{dx}{dt} = P(t, I(t))$ (P decreasing in I)

$$i(x+\delta x, t+\delta t) = i(x,t) - \mu_i\, i(x,t)\,\delta t$$

$$\Rightarrow \boxed{\frac{\partial i}{\partial t} + P(t, I(t))\frac{\partial i}{\partial x} = -\mu_i\, i(x,t)} \qquad x \in (0,l),\ t > 0.$$

Also
$$\frac{dI}{dt} = \int_0^l \left(-P(t,I(t))\frac{\partial i}{\partial x} - \mu_i\, i(x,t)\right)dx$$
$$= P(t, I(t))\left(i(0,t) - i(l,t)\right) - \mu_i\, I(t).$$

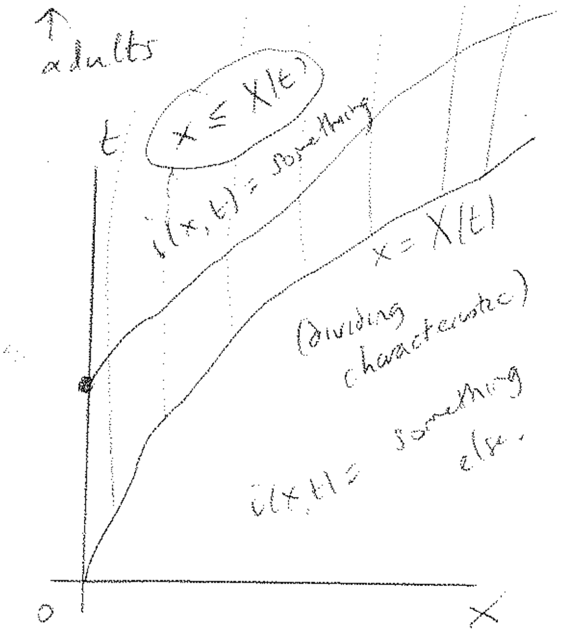
$$P(t, I(t))\, i(0,t) = \text{birth rate} \overset{\text{assume}}{=} \underset{\uparrow \text{ adults}}{B(A(t))}$$

$$P(t, I(t))\, \underbrace{i(l,t)}_{\text{need to calculate}} = \text{maturation rate.}$$

define $X(t) = \displaystyle\int_0^t P(\xi, I(\xi))\,d\xi$

Can show, for $x \le X(t)$, that
$$i(x,t) = i(0, t - T(x,t))\, e^{-\mu_i T(x,t)}$$

where $\displaystyle\int_{t-T(x,t)}^{t} P(\xi, I(\xi))\,d\xi = x.$



larvae mature on reaching size l. Let $\tau(t) = T(l,t)$

$$\boxed{\int_{t-\tau(t)}^{t} P(\xi, I(\xi))\,d\xi = l}$$

threshold condition.

$$\frac{dI}{dt} = B(A(t)) - P(t, I(t))\, i(l,t) - \mu_i I(t)$$

$$= B(A(t)) - P(t, I(t))\, i(0, t-\tau(t))\, e^{-\mu_i \tau(t)} - \mu_i I(t)$$

$$\boxed{\frac{dI(t)}{dt} = B(A(t)) - \frac{P(t, I(t))\, B(A(t-\tau(t)))}{P(t-\tau(t), I(t-\tau(t)))} e^{-\mu_i \tau(t)} - \mu_i I(t)}$$

$$\boxed{\frac{dA(t)}{dt} = \frac{P(t, I(t))\, B(A(t-\tau(t)))}{P(t-\tau(t), I(t-\tau(t)))} e^{-\mu_i \tau(t)} - \mu_a A(t)}$$

Conversion to constant delay. Suppose $P(t, I) = P_0(I)$, & let

$$x = \int_0^t P_0(I(\xi))\,d\xi$$

(new size-like indep. variable).

$$x \leftrightarrow t$$

$$x - l \leftrightarrow t - \tau(t)$$

model becomes

$$\frac{dA(x)}{dx} = -\mu_a \frac{A(x)}{P_0(I(x))} + e^{-\mu_i \tau_0(I_x)} \frac{B(A(x-l))}{P_0(I(x-l))}$$

$$\frac{dI(x)}{dx} = -\mu_i \frac{I(x)}{P_0(I(x))} + \frac{B(A(x))}{P_0(I(x))} - e^{-\mu_i \tau_0(I_x)} \frac{B(A(x-l))}{P_0(I(x-l))}$$

$$\tau_0(\phi) = \int_{-l}^{0} \frac{d\bar{x}}{P_0(\phi(\bar{x}))}$$

$$I_x(\bar{x}) = I(x + \bar{x})$$

linear stability of an equilibrium

Steady state (A^*, I^*)

in linearised system, $\tau(t)$ replaced by $\dfrac{\ell}{P_0(I^*)}$

characteristic equation:

$$\frac{P_0(I^*)}{\ell}(\Lambda + \hat{\mu}_a) - B'(A^*)\,e^{-(\Lambda+\hat{\mu}_i)} = \frac{\varepsilon B'(A^*)(\Lambda + \hat{\mu}_i)\,k(\Lambda)\,k(\Lambda + \hat{\mu}_i)}{1 + \varepsilon k(\Lambda)}$$

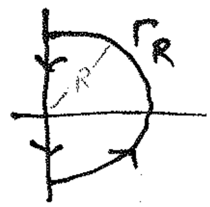
holomorphic in $\mathrm{Re}\,\Lambda \geq 0$ if ε [illegible]

$$\Lambda = \lambda \frac{\ell}{P_0(I^*)}, \qquad (A(t), I(t)) = (A^*, I^*) + \underline{c}\, e^{\lambda t}$$

$$k(x) = \frac{1 - e^{-x}}{x}$$

$\varepsilon = 0$ if all larvae take same time to mature.

Techniques:



* Rouches theorem
* $N_0(f) - N_p(f) = \dfrac{1}{2\pi}\Big[\arg f(z)\Big]_{z \in \Gamma_R}$

Allee effect

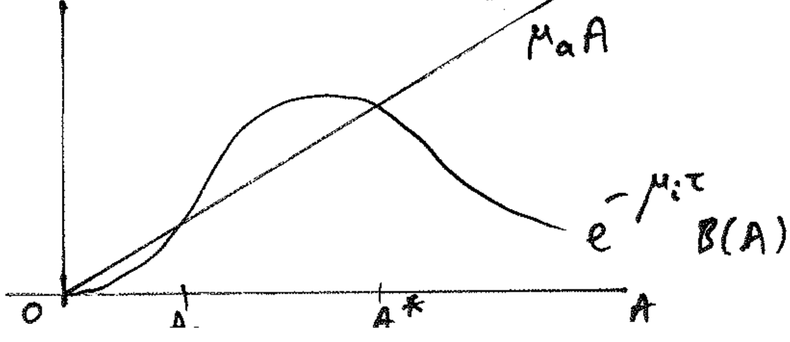
If
* Allee effect, & $P(I) = \begin{cases} P_0 & I \leq I_c \\ 0 & I > I_c \end{cases}$ pausing of larval developmen when too much competition
* μ_i small, μ_a large (+ other conds)
* may have $A(t) \to 0$ even if starts off large

Starting point: constant delay case:

$$\frac{dA}{dt} = -\mu_a A(t) + e^{-\mu_i \tau} B(A(t-\tau))$$

with:

(A_c = Allee effect threshold)



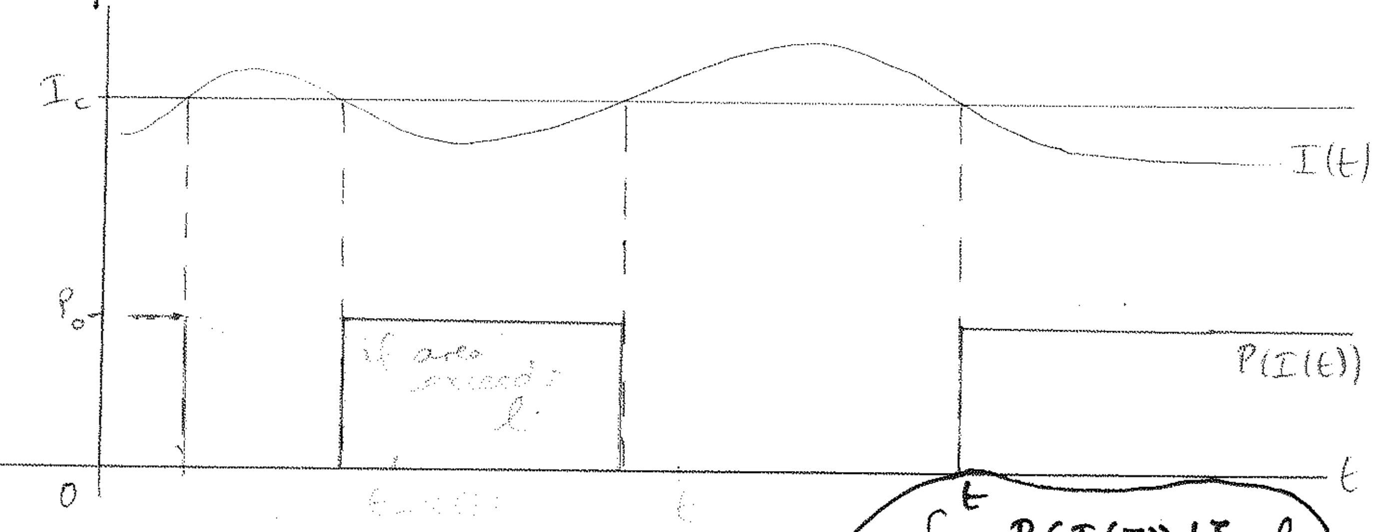
lemma

If $\exists\, t^*$ s.t. $A(t) \in [0, A_c)$ $\forall$ $t \in [t^*, t^* + \tau]$

then $A(t) \to 0$ as $t \to \infty$.

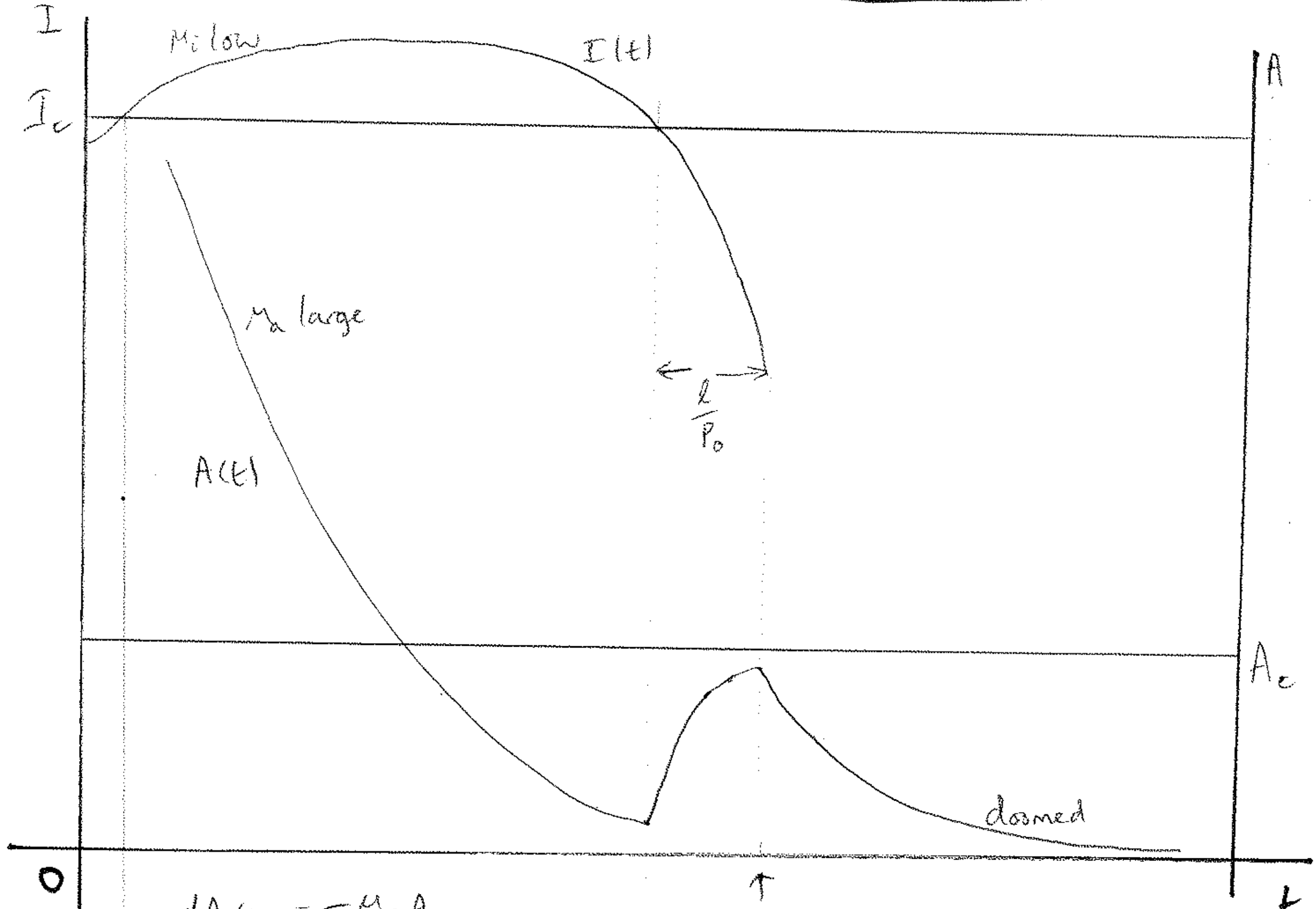
lemma $\nexists\, t$ s.t. $P(I(t-\tau(t))) = 0$.

Proof



$$\int_{t-\tau(t)}^{t} P(I(\xi))\,d\xi = l$$

extinction with Allee effect



$dA/dt = -\mu_a A$

$$\frac{dI}{dt} = -\mu_i I + B(A)$$

delay constant from now on