Asymptotics of (SWE-1) and (SWE-2) in the shallow-water limit in three-dimensions

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— 16 July 2010 —

This technical report is based on §7.3 in [2]. It shows how the principal assumptions in [2] are satisfied in the standard shallow-water limit. It is extracted here for readers only interested in the asymptotic argument.

The conditions (SWE-1) and (SWE-2) are global. That is, there is no particular restriction on parameter values. Indeed they may be satisfied even in deep water. However, the most natural regime where one would expect them to be satisfied is in the shallow-water regime. In this report a scaling argument and asymptotics are used to analyze (SWE-1) and (SWE-2) in the shallow-water limit. Here only the simplest scaling is considered. The small parameter representing shallow water is

$$\varepsilon = \frac{h_0}{L} , \tag{1}$$

where L is a representative horizontal length scale. Let $U_0 = \sqrt{gh_0}$ be the representative horizontal velocity scale. Introduce the standard shallow-water scaling (e.g. p. 482 of [3]),

$$\begin{aligned} \widetilde{x} &= \frac{x}{L}, \quad \widetilde{y} = \frac{y}{L}, \quad \widetilde{z} = \frac{z}{h_0} = \frac{z}{\varepsilon L}, \quad \widetilde{t} = \frac{tU_0}{L}, \\ \widetilde{u} &= \frac{u}{U_0}, \quad \widetilde{v} = \frac{v}{U_0}, \quad \widetilde{w} = \frac{w}{\varepsilon U_0}, \quad \widetilde{h} = \frac{h}{h_0}. \end{aligned} \tag{2}$$

The scaled version of the surface velocities are denoted by $\widetilde{U}$, $\widetilde{V}$ and $\widetilde{W}$.

The typical strategy for deriving an asymptotic shallow-water model is to scale the full Euler equations, and then use an asymptotic argument to reduce the vertical pressure field and vertical velocities (e.g. §5.1 of [3]). Here however we have an advantage as the full Euler equations have been reduced to the exact surface equations (7.1) in [2]. Hence the strategy here is to start by scaling the exact surface equations, and then apply an asymptotic argument.

To check (SWE-1), start by scaling the exact mass equation

$$\widetilde{h}_{\widetilde{t}} + (\widetilde{h}\widetilde{U})_{\widetilde{x}} + (\widetilde{h}\widetilde{V})_{\widetilde{y}} = \widetilde{W} + \widetilde{h}(\widetilde{U}_{\widetilde{x}} + \widetilde{V}_{\widetilde{y}}) . \tag{3}$$

At first glance it appears that the left-hand side and the right-hand side are of the same order, since ε does not appear. However, the *sum* on the right-hand side is of higher order. The fact that the right-hand side is of higher order is intuitively clear, since it can be expressed in terms of the velocity differences $U-\overline{u}$ and $V-\overline{v}$, and in the shallow-water approximation the horizontal surface velocites (U,V) and vertically-averaged horizontal velocities $(\overline{u},\overline{v})$ are asymptotically equivalent. However, to make this precise we need to bring in the vorticity field.

Go back to the unscaled mass equation and rewrite the right-hand side using

$$(h(U-\overline{u}))_x + (h(V-\overline{v}))_y = W + h(U_x+V_y) = h_t + (hU)_x + (hV)_y\,, \tag{4}$$

giving

$$h_t + (hU)_x + (hV)_y = \frac{\partial}{\partial x}\left(\int_0^h z u_z\,\mathrm{d}z\right) + \frac{\partial}{\partial y}\left(\int_0^h z v_z\,\mathrm{d}z\right)\,.$$

Substitute for u_z and v_z using the vorticity field

$$\mathcal{V} = (\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3) := \nabla\times\mathbf{u}\,.$$

giving

$$h_t + (hU)_x + (hV)_y = \frac{\partial}{\partial x}\left(\int_0^h z\left(\mathcal{V}_2 + w_x\right)\,\mathrm{d}z\right) + \frac{\partial}{\partial y}\left(\int_0^h z\left(w_y - \mathcal{V}_1\right)\,\mathrm{d}z\right)\,. \tag{5}$$

This equation is exact. The key to showing the right-hand side is of higher order is the scaling of the vorticity. The appropriate scaling is to assume that *the vorticity is asymptotically vertical*,

$$(\mathcal{V}_1,\mathcal{V}_2,\mathcal{V}_3) = \frac{U_0}{L}\left(\varepsilon\widetilde{\mathcal{V}}_1, \varepsilon\widetilde{\mathcal{V}}_2, \widetilde{\mathcal{V}}_3\right)\,. \tag{6}$$

This property of vorticity is implicit in the classical shallow-water theory, and here it is made explicit.

Scaling (5) then gives

$$\widetilde{h}_{\widetilde{t}} + (\widetilde{h}\widetilde{U})_{\widetilde{x}} + (\widetilde{h}\widetilde{V})_{\widetilde{y}} = \varepsilon^2\Delta(x,y,t,\varepsilon)\,. \tag{7}$$

where

$$\Delta = \frac{\partial}{\partial\widetilde{x}}\int_0^{\widetilde{h}} \widetilde{z}\left(\widetilde{\mathcal{V}}_2 + \frac{\partial\widetilde{w}}{\partial\widetilde{x}}\right)\,\mathrm{d}\widetilde{z} + \frac{\partial}{\partial\widetilde{y}}\int_0^{\widetilde{h}} \widetilde{z}\left(-\widetilde{\mathcal{V}}_1 + \frac{\partial\widetilde{w}}{\partial\widetilde{y}}\right)\,\mathrm{d}\widetilde{z}\,. \tag{8}$$

Taking the limit $\varepsilon\to 0$ shows that (SWE-1) is satisfied. However, to be precise it is essential that

$$\Delta(x,y,t,\varepsilon) \quad \text{is bounded in the limit } \varepsilon\to 0\,. \tag{9}$$

Assumption (SWE-2) requires that the vertical acceleration in the two terms

$$\left(a_{11} + \frac{Dw}{Dt}\bigg|^h\right) \quad \text{and} \quad \left(a_{22} + \frac{Dw}{Dt}\bigg|^h\right)\,. \tag{10}$$

in (7.1) be small, relative to magnitude of a_{11} and a_{22}. After scaling, the Lagrangian vertical acceleration in the interior becomes

$$\frac{Dw}{Dt} = \varepsilon\frac{U_0^2}{L}\left(\frac{\partial\widetilde{w}}{\partial\widetilde{t}} + \widetilde{u}\frac{\partial\widetilde{w}}{\partial\widetilde{x}} + \widetilde{v}\frac{\partial\widetilde{w}}{\partial\widetilde{y}} + \widetilde{w}\frac{\partial\widetilde{w}}{\partial\widetilde{z}}\right) := \varepsilon\frac{U_0^2}{L}\frac{D\widetilde{w}}{D\widetilde{t}}\,.$$

Hence

$$\left.\frac{Dw}{Dt}\right|^{h} = \varepsilon\frac{U_0^2}{L}\left.\frac{D\widetilde{w}}{D\widetilde{t}}\right|^{\widetilde{h}} = g\varepsilon^2 \left.\frac{D\widetilde{w}}{D\widetilde{t}}\right|^{\widetilde{h}} ,$$

using $U_0^2 = gh_0 = gL\varepsilon$. The scaled version of the first term in (10) is therefore

$$\left(a_{11} + \left.\frac{Dw}{Dt}\right|^{h}\right) = g\left(\frac{a_{11}}{g} + \varepsilon^2\left.\frac{D\widetilde{w}}{D\widetilde{t}}\right|^{\widetilde{h}}\right) ,$$

with a similar expression for the a_{22} term.

In the shallow-water regime, the assumption (SWE-2) is satisfied if

$$\frac{a_{11}}{g} \quad \text{and} \quad \frac{a_{22}}{g} \quad \text{are of order one and} \quad \left|\frac{D\widetilde{w}}{D\widetilde{t}}\right|^{\widetilde{h}}\Big| \quad \text{is bounded as} \quad \varepsilon\to 0\,. \tag{11}$$

However, by introducing scaling and taking an asymptotic limit, other anomalies can be introduced. We have to ensure that b_1 and b_2 are of the same order – or of higher order – as the left-hand side of the second and third equations of (7.1) in [2]. Look at the second equation with surface tension neglected

$$U_t + UU_x + VU_y + \left(a_{11} + \left.\frac{Dw}{Dt}\right|^{h}\right) h_x + a_{12}\, h_y = b_1\,.$$

The left-hand side scales like $U_0^2/L = g\varepsilon$. With the standard scaling for $\mathbf{\Omega}$,

$$(\Omega_1,\Omega_2,\Omega_3) = \frac{U_0}{L}(\widetilde{\Omega}_1,\widetilde{\Omega}_2,\widetilde{\Omega}_3)\,,$$

all the terms in b_1 (see [2]) for the definition of b_1) are of order ε or higher except for the term $g\mathbf{Q}\mathbf{e}_1\cdot\mathbf{e}_3$ which is of order unity. In scaled variables it will be of order ε^{-1}. Hence this scaling puts a restriction on the angular velocity. A natural scaling that renders b_1 consistent is to take the angular velocity to be asymptotically vertical, like the vorticity,

$$(\Omega_1,\Omega_2,\Omega_3) = \frac{U_0}{L}(\varepsilon\widetilde{\Omega}_1,\varepsilon\widetilde{\Omega}_2,\widetilde{\Omega}_3)\,, \tag{12}$$

To verify that b_1 is now consistent it is necessary to show that

$$\lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\mathbf{Q}(\widetilde{t},\varepsilon)\mathbf{e}_1\cdot\mathbf{e}_3 \quad \text{is of order unity (or higher in } \varepsilon)\,.$$

This property follows from the scaling (12). In scaled variables, $\mathbf{Q}(t,\varepsilon)$ satisfies

$$\frac{d}{d\widetilde{t}}\mathbf{Q} = \mathbf{Q}\widetilde{\widetilde{\mathbf{\Omega}}}\,, \quad \widetilde{\widetilde{\mathbf{\Omega}}} = \begin{bmatrix} 0 & -\widetilde{\Omega}_3 & \varepsilon\widetilde{\Omega}_2 \\ \widetilde{\Omega}_3 & 0 & -\varepsilon\widetilde{\Omega}_1 \\ -\varepsilon\widetilde{\Omega}_2 & \varepsilon\widetilde{\Omega}_1 & 0 \end{bmatrix}\,.$$

This expression in unscaled variables is just the definition of the *body* angular velocity (see equation(2.1) in [2]). Hence, in the limit as $\varepsilon \to 0$,

$$\lim_{\varepsilon \to 0} \mathbf{Q}(\widetilde{t}, \varepsilon) := \mathbf{Q}(\widetilde{t}, 0) = \begin{bmatrix} \cos\psi(\widetilde{t}) & -\sin\psi(\widetilde{t}) & 0 \\ \sin\psi(\widetilde{t}) & \cos\psi(\widetilde{t}) & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{where} \quad \frac{d\psi}{d\widetilde{t}} = \widetilde{\Omega}_3 \,,$$

and so clearly

$$\mathbf{Q}(\widetilde{t}, 0)\mathbf{e}_1 \cdot \mathbf{e}_3 = 0 \,,$$

confirming that $\mathbf{Q}(\widetilde{t}, \varepsilon)\mathbf{e}_1 \cdot \mathbf{e}_3 = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$. A similar argument shows that the term $\mathbf{Q}(\widetilde{t}, \varepsilon)\mathbf{e}_2 \cdot \mathbf{e}_3 = \mathcal{O}(\varepsilon)$ as $\varepsilon \to 0$, which appears in b_2.

The above scaling is only one of many, even in the shallow-water limit. A study of the various asymptotic regimes is outside the scope of this report. Our main guide is the two meta-assumptions (SWE-1) and (SWE-2). They are required in general and will have to be satisfied by any choice of scaling.

On the other hand, the above shallow-water scaling does appear implicitly in the numerical results reported in [2]) and on the website [1]. We have found that roll-pitch type forcing (i.e. Ω_1 and Ω_2 nonzero) requires very small amplitude in order to avoid large fluid motions that would violate (SWE-1) and/or (SWE-2), whereas the amplitude of yaw (Ω_3 nonzero) can be much larger (e.g. §13 in [2]) and similarly the amplitude of translation ($\mathbf{q}$) can be of order unity (e.g. the results on the London Eye in §15 of [2]).

References

[1] `http://personal.maths.surrey.ac.uk/st/T.Bridges/SLOSH/`

[2] H. Alemi Ardakani & T.J. Bridges. *Shallow-water sloshing in vessels undergoing prescribed rigid-body motion in three dimensions*, J. Fluid Mech. (*sub judice*) (2010).

[3] M.W. Dingemans. *Water Wave Propagation Over Uneven Bottoms. Part 2: No nlinear Wave Propagation*, World Scientific: Singapore (1997).