GEOMETRICAL TOOLS FOR PDES ON A SURFACE – WITH ROTATING SHALLOW WATER EQUATIONS ON A SPHERE

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ABSTRACT. This is an excerpt from my paper with A. Mahalov [1]. PDE theories with Riemannian geometry are long studied subject — c.f. for example the texts cited in the bibliography. Here, I only present a very brief and elementary explanation, mainly about differential operators on a 2D surface. Generalization to higher dimension needs more caution. The last section on spherical rotating shallow water equations is also motivated by [5] and my interest in geophysical applications.

1. General Theory

Let $\mathbb{M}$ denote a 2-dimensional, compact, Riemannian manifold without boundary, typically the unit sphere $\mathbb{M} = \mathbb{S}^{3-1}$, endowed with metric $\mathfrak{g}$ inherited from the embedding Euclidean space $\mathbb{R}^3$. Let $p \in \mathbb{M}$ denote a point with local coordinates (p_1, p_2).

Any vector field $\mathbf{u}$ in the tangent bundle $T\mathbb{M}$ is identified with a field of *directional differentials*, which is written in local coordinates as

$$\mathbf{u} = \sum_i a^i \frac{\partial}{\partial p_i}.$$

We use the notation

$$\nabla_{\mathbf{u}} f := \sum_i a^i \frac{\partial f}{\partial p_i} \tag{1.1}$$

to denote the directional derivative of a scalar-valued function f in the direction of $\mathbf{u}$. Using the orthogonal projection $\underset{T\mathbb{R}^3 \to T\mathbb{M}}{\text{Proj}}$ induced by the Euclidean metric of $\mathbb{R}^3$, we define the covariant derivative of a vector field $\mathbf{v} \in T\mathbb{M}$ along another vector field $\mathbf{u} \in T\mathbb{M}$,

$$\nabla_{\mathbf{u}} \mathbf{v} := \underset{T\mathbb{R}^3 \to T\mathbb{M}}{\text{Proj}} \sum_{i=1}^{3} (\nabla_{\mathbf{u}} v_i) \mathbf{e}_i. \tag{1.2}$$

Here, $\mathbf{v}$ is expressed in an orthonormal basis of $\mathbb{R}^3$ as $\mathbf{v} = v_1\mathbf{e}_1 + v_2\mathbf{e}_2 + v_3\mathbf{e}_3$.

The metric $\mathfrak{g}$ is identified with a $(0, 2)$ tensor, simply put, an 2×2 matrix $(g_{ij})_{2\times 2}$ in local coordinates. Thus, the vector inner product follows

$$\mathfrak{g}\left(\frac{\partial}{\partial p_i}, \frac{\partial}{\partial p_j}\right) = g_{ij} \quad \text{for } 1 \le i, j \le 2.$$

The definitions and basic properties of *differential forms* can be found in e.g. [4]. We only sketch the following facts that will be used in this section. A differential k-form β, at given point $p \in \mathbb{M}$, maps any k-tuple of tangent vectors to a scalar. In particular, a 0-form is identified with a scalar-valued function. The 1-form dp_i in local coordinates satisfies $dp_i(\frac{\partial}{\partial p_j}) = \delta_{ij}$. The *exterior*

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differential d maps a k-form to a $(k+1)$form. For example, for 0-form f, $df = \sum_i \frac{\partial f}{\partial p_j} dp_i$ so that $df(\mathbf{u}) = \nabla_{\mathbf{u}} f$. The *wedge product* of a k-form α and l-form β, denoted by $\alpha \wedge \beta$, is a $(k+l)$ form . It is skew-commutative in the sense that $\alpha \wedge \beta = (-1)^{kl} \beta \wedge \alpha$.

1.1. **Hodge Theory.** ([4, 3])

The Hodge *-operator, defined in an orthonormal basis [1] $\frac{\partial}{\partial p_1}, \frac{\partial}{\partial p_2}$ in a sub-region of $\mathbb{M}$, satisfies

$$*dp_1 = dp_2, \quad *dp_2 = -dp_1, \quad *1 = dp_1 \wedge dp_2, \quad *(dp_1 \wedge dp_2) = 1.$$

It is easy to see that Hodge *-operator maps between k-forms and $(n-k)$form. And its square, $**$ amounts to identity or (-1) times identity.

Using the Hodge star operator, we define the co-differential for any k-forms α in an n-dimensional manifold,

$$\textbf{codifferential}: \quad \delta\alpha := (-1)^k *^{-1} d * \alpha = (-1)^{n(k+1)+1} * d * \alpha,$$

and in particular, for $n = 2$,

$$\delta\alpha = - * d * \alpha.$$

So, δ maps a k-form to a $(k-1)$form.

The Hodge Laplacian (a.k.a. Laplace-Beltrami operator and Laplace-de Rham operator) is then defined by

$$\Delta_H := d\delta + \delta d. \tag{1.3}$$

In particular, for a scalar-valued function f in a local basis $\left\{\frac{\partial}{\partial p_i}\right\}$ with metric $\mathfrak{g}$, it is identified as

$$\Delta_H f = -\frac{1}{\sqrt{|\mathfrak{g}|}} \sum_{i,j} \partial_i (\sqrt{|\mathfrak{g}|} g^{ij} \partial_j f)$$

where (g^{ij}) is the matrix inverse of (g_{ij}). Thus, on a surface $\mathbb{M}$, the Hodge Laplacian Δ_H defined in (1.3) amounts to the surface Laplacian $\Delta_{\mathbb{M}}$ times (-1). In particular, if $\mathbb{M}$ is a two-dimensional surface, then

$$\text{for scalar function } f, \quad \Delta_{\mathbb{M}} f = -\delta df = *d * df \tag{1.4}$$

since $\delta f = 0$ for a 0-form f. For consistency, we also fix the surface Laplacian $\Delta_{\mathbb{M}}$ of 1-forms as the Hodge Laplacian Δ_H times (-1),

$$\text{for 1-form } \alpha, \quad \Delta_{\mathbb{M}} \alpha = -(d\delta + \delta d)\alpha = (d * d * + * d * d)\alpha \tag{1.5}$$

For now on, we will use Δ for $\Delta_{\mathbb{M}}$.

The **Hodge decomposition** theorem in its most general form states that for any k-form ω on an oriented compact Riemannian manifold, there exist a $(k-1)$-form α, $(k+1)$-form β and a harmonic k-form γ satisfying $\Delta_H \gamma = 0$, s.t.

$$\omega = d\alpha + \delta\beta + \gamma.$$

In particular, for any 1-form ω on a 2-dimensional manifold with the 1st Betti number 0 (loosely speaking, there is no "holes"), there exist two scalar-valued functions Φ, Ψ such that

$$\omega = d\Phi + \delta(*\Psi) = d\Phi - *d\Psi. \tag{1.6}$$

[1] The existence of such basis is guaranteed by the Gram-Schmidt orthogonalization process.

Here, the third term drops out of the RHS of (1.6) because, by the Hodge theory, the dimension of the space of harmonic k-forms on M equals the k-th Betti number of M. In the cohomology class containing the unit sphere $\mathbb{S}^2$, the 0th, 1st and 2nd Betti numbers are respectively $1, 0, 1$. In other words, the only harmonic 1-form on $\mathbb{S}^2$ is zero.

1.2. **In connection with vector fields.** In a Riemannian manifold, there is a 1-to-1 correspondence, induced by the metric $\mathfrak{g}$, between vectors and 1-forms. They are the so called "musical isomorphisms" demoted by $\flat$ and $\sharp$. For any vector fields $\mathbf{u}, \mathbf{v}$, the 1-form $\mathbf{u}^\flat$ satisifies,

$$\mathbf{u}^\flat(\mathbf{v}) = \mathfrak{g}(\mathbf{u}, \mathbf{v}), \qquad (\mathbf{u}^\flat)^\sharp = \mathbf{u}.$$

In a (local) orthonormal basis, $\flat$ and $\sharp$ map between vectors and 1-forms with identical coordinates.

In a 2-dimensional Riemannian manifold, in order to define the divergence and curl of a vector field $\mathbf{u} \in T\mathbb{M}$, we use $\flat$ to map it to 1-form and then apply δ and d to obtain the scalar fields[2],

$$\textbf{divergence}, \quad \operatorname{div}\mathbf{u} := -\delta(\mathbf{u}^\flat) = *d*(\mathbf{u}^\flat) \tag{1.7}$$

$$\textbf{curl}, \quad \operatorname{curl}\mathbf{u} := -\delta(*\mathbf{u}^\flat) = -*d(\mathbf{u}^\flat) \tag{1.8}$$

For a scalar field f, we define gradient and its $\pi/2$ rotation as

$$\textbf{gradient}, \quad \nabla f := (df)^\sharp \tag{1.9}$$

$$\textbf{rotated gradient}, \quad \nabla^\perp f := -(*df)^\sharp \tag{1.10}$$

We also define the counterclockwise $\pi/2$ rotation operator $^\perp$ acting on a vector field as

$$\mathbf{u}^\perp := -(*\mathbf{u}^\flat)^\sharp \tag{1.11}$$

so that, consistently, $\nabla^\perp f = (\nabla f)^\perp$ and $\operatorname{div}\mathbf{u} = \operatorname{curl}\mathbf{u}^\perp$.

Combine these definitions with that of the surface Laplacian (1.4), (1.5) to obtain

$$\textbf{Laplacian} \text{ of scalar}, \quad \Delta f := \operatorname{div}\nabla f = \operatorname{curl}\nabla^\perp f \tag{1.12}$$

$$\textbf{Laplacian} \text{ of vector}, \quad \Delta\mathbf{u} := \nabla\operatorname{div}\mathbf{u} + \nabla^\perp\operatorname{curl}\mathbf{u}. \tag{1.13}$$

An immediate consequence is that

$$\Delta \text{ commutes with each one of } \operatorname{div}, \operatorname{curl}, \nabla, \nabla^\perp. \tag{1.14}$$

To this end, the vector-field version of Hodge decomposition (1.6) becomes, for smooth vector field $\mathbf{u}$ on $\mathbb{S}^2$, there exist scalar fields Φ, Ψ, s.t.

$$\textbf{Hodge decomposition}, \quad \mathbf{u} = \nabla\Phi + \nabla^\perp\Psi. \tag{1.15}$$

We note that, by the virtue of (1.12), the decomposition satisfies

$$\operatorname{div}\mathbf{u} = \Delta\Phi, \qquad \operatorname{curl}\mathbf{u} = \Delta\Psi.$$

It is also easy to use these definitions to verify the following properties,

$$\operatorname{curl}\nabla f = \operatorname{div}\nabla^\perp f = 0 \tag{1.16}$$

due to $dd = 0$ and $\delta\delta = 0$; and,

$$\operatorname{curl}(f\mathbf{u}^\perp) = \operatorname{div}(f\mathbf{u}) = \nabla f \cdot \mathbf{u} + f\operatorname{div}\mathbf{u}, \tag{1.17}$$

[2]Since 0-forms are identified with scalar fields, we use Hodge *-operator to map between 0-forms and 2-forms

as a consequence of the product rule for differential d.

1.3. **In connection with surface integrals.** The integral of scalar field f over an n-dimensional, oriented Riemannian manifold $\mathbb{M}$, defined in differential-geometric terms (e.g. [4, Section 4.10]), is the integral of the n-form $*f$

$$\int_{\mathbb{M}} f = \int_{\mathbb{M}} *f. \tag{1.18}$$

Note that any n-form defines a measure on $\mathbb{M}$.

For the most general case, the inner product of k-forms α_1, α_2 is defined as

$$\langle \alpha_1, \alpha_2 \rangle := \int_{\mathbb{M}} (\alpha_1 \wedge *\alpha_2).$$

In particular, for vector fields $\mathbf{u}, \mathbf{v}$, we map them to 1-forms and define $L^2(\mathbb{M})$ inner product as

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(\mathbb{M})} := \langle \mathbf{u}^\flat, \mathbf{v}^\flat \rangle.$$

This coincides exactly with the more conventional definition

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(\mathbb{M})} := \int_{\mathbb{M}} \mathfrak{g}(\mathbf{u}, \mathbf{v})$$

where $\mathfrak{g}$ is the given metric — in Euclidean geometry, $\mathfrak{g}(\mathbf{u}, \mathbf{v}) = \mathbf{u} \cdot \mathbf{v}$.

It then follows from the celebrated Stokes theorem (e.g. [4]) that, in the case when $\mathbb{M}$ has no boundary, the codifferential is the adjoint of exterior differential w.r.t. $L^2(\mathbb{M})$ inner product

$$\langle d\alpha, \beta \rangle = \langle \alpha, \delta\beta \rangle.$$

This duality relation, together with definitions (1.7) – (1.13), immediately leads to the following "integrating-by-parts" formulas on a surface $\mathbb{M}$, which generalize the Green's identities and give adjoint relations of the differential operators we just defined.

$$\langle \nabla f, \mathbf{u} \rangle = -\langle f, \operatorname{div} \mathbf{u} \rangle, \tag{1.19}$$

$$\langle \nabla^\perp f, \mathbf{u} \rangle = -\langle f, \operatorname{curl} \mathbf{u} \rangle, \tag{1.20}$$

$$\langle f, \Delta h \rangle = \langle \Delta f, h \rangle = -\langle \nabla f, \nabla h \rangle, \tag{1.21}$$

$$\langle \mathbf{u}, \Delta \mathbf{v} \rangle = \langle \Delta \mathbf{u}, \mathbf{v} \rangle = -\langle \operatorname{div} \mathbf{u}, \operatorname{div} \mathbf{v} \rangle - \langle \operatorname{curl} \mathbf{u}, \operatorname{curl} \mathbf{v} \rangle. \tag{1.22}$$

Here, for simplicity, the $L^2(\mathbb{M})$ subscript is omitted from all $\langle , \rangle$ expressions. The $\langle , \rangle$ of scalar fields is also easily understood.

2. Spherical coordinates on $\mathbb{M} = \mathbb{S}^2$.

Although the proofs in this article are independent of any local coordinate systems, we provide here, for interested readers, the spherical-coordinate forms of some of the differential operators defined above.

Let λ denote the longitude and θ the latitude with $\theta = \pi/2$ at North Pole. Let $\mathbf{i}$, $\mathbf{j}$ denote the unit tangent vectors in the increasing directions of λ and θ. Then, at point p away from the poles,

$$\partial_\lambda = \cos\theta\, \mathbf{i}, \qquad \partial_\theta = \mathbf{j},$$

namely,

$$\frac{1}{\cos\theta}\partial_\lambda \quad \text{and} \quad \partial_\theta \qquad \text{form an orthonormal basis of } T\mathbb{M}_p.$$

Therefore, the musical isomorphisms, in λ, θ coordinates, satisfy

$$(\frac{1}{\cos\theta}\partial_\lambda)^\flat = \cos\theta d\lambda \quad \text{and} \quad (\partial_\theta)^\flat = d\theta \quad \text{form an orthonormal basis of } T^*\mathbb{M}_p.$$

In this context, the Hodge *-operator satisfies, with smooth scalar fields f_1, f_2, f,

$$\text{for 1-forms,} \qquad *\,(f_1 d\lambda + f_2 d\theta) = \frac{f_1}{\cos\theta} d\theta - f_2 \cos\theta d\lambda$$

$$\text{for 0-forms and 2-forms,} \qquad *\,(f d\lambda \wedge d\theta) = \frac{f}{\cos\theta}, \qquad *f = f\cos\theta d\lambda \wedge d\theta$$

Combining the last equation with (1.18), we have the spherical expression for the integral of scalar field f over $\mathbb{S}^2$,

$$\int_{\mathbb{S}^2} f = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} f(\lambda, \theta) \cos\theta \, d\lambda d\theta,$$

and inner product of vector fields, $\mathbf{u} = u_1 \,\mathbf{i} + u_2 \,\mathbf{j}$, $\mathbf{v} = v_1 \,\mathbf{i} + v_2 \,\mathbf{j}$,

$$\langle \mathbf{u}, \mathbf{v} \rangle_{L^2(\mathbb{S}^2)} = \int_{-\pi/2}^{\pi/2} \int_{-\pi}^{\pi} (u_1 v_1 + u_2 v_2 \,) \cos\theta \, d\lambda d\theta$$

The differential operators defined in (1.7) — (1.13) then become,

for scalar field f

$$\begin{aligned} \nabla f =& \frac{\mathbf{i}}{\cos\theta} \partial_\lambda f + \mathbf{j}\,\partial_\theta f \\ \nabla^\perp f =& \mathbf{i}\,\partial_\theta f \frac{\mathbf{j}}{\cos\theta} \partial_\lambda f \\ \Delta f =& \frac{1}{\cos^2\theta} (\partial_\lambda^2 f + \cos\theta \partial_\theta (\cos\theta \partial_\theta f)), \end{aligned}$$

and

$$\begin{aligned} \text{for vector field} \qquad \mathbf{u} =& u_1 \,\mathbf{i} + u_2 \,\mathbf{j} \\ \operatorname{div} \mathbf{u} =& \frac{1}{\cos\theta} (\partial_\lambda u_1 + \partial_\theta (u_2 \cos\theta)) \\ \operatorname{curl} \mathbf{u} =& \frac{1}{\cos\theta} (\partial_\lambda u_2 - \partial_\theta (u_1 \cos\theta)). \end{aligned}$$

The surface Laplacian of $\mathbf{u}$ can also be expressed using (1.13) and the formulas above.

The directional derivative of a scalar, (1.1), can be expressed as

$$\nabla_{\mathbf{u}} f = \mathbf{u} \cdot \nabla f = \frac{u_1}{\cos\theta} \partial_\lambda f + u_2 \partial_\theta f.$$

For the covariant derivative $\nabla_{\mathbf{u}} \mathbf{v}$, we combine the above formula with (1.2). First, we calculate in $\mathbb{R}^3$

$$\begin{aligned} \mathbf{u} \cdot \nabla \,\mathbf{i} \; =& \frac{u_1}{\cos\theta} \partial_\lambda \,\mathbf{i} \, + u_2 \partial_\theta \,\mathbf{i} \; = \frac{u_1}{\cos\theta} (\sin\theta \,\mathbf{j} \, + \cos\theta \,\mathbf{k} \,), \\ \mathbf{u} \cdot \nabla \,\mathbf{j} \; =& \frac{u_1}{\cos\theta} \partial_\lambda \,\mathbf{j} \, + u_2 \partial_\theta \,\mathbf{j} \\ =& \frac{u_1}{\cos\theta} \partial_\lambda [\,\mathbf{j} \, - (\,\mathbf{j} \cdot \mathbf{z}\,)\,\mathbf{z}\,] + u_2 \partial_\theta \,\mathbf{j} \\ =& - \frac{u_1}{\cos\theta} \sin\theta \,\mathbf{i} \, - u_2 \,\mathbf{k} \end{aligned}$$

where $\mathbf{k}$ is the unit outward normal to $\mathbb{S}^2$, and $\mathbf{z}$ is the third unit vector in the cartesian basis. Thus, by (1.2), we have the covariant derivatives on $\mathbb{S}^2$,

$$\nabla_{\mathbf{u}}\mathbf{i} = u_1 \tan\theta\, \mathbf{j}, \quad \nabla_{\mathbf{u}}\mathbf{j} = -u_1 \tan\theta\, \mathbf{i}.$$

Finally, apply the product rule

$$\nabla_{\mathbf{u}}\mathbf{v} = \nabla_{\mathbf{u}}(v_1\,\mathbf{i} + v_2\,\mathbf{j}) = (\mathbf{u}\cdot\nabla v_1)\,\mathbf{i} + (\mathbf{u}\cdot\nabla v_2)\,\mathbf{j} + (u_1 \tan\theta)(\mathbf{k} \times \mathbf{v}). \tag{2.1}$$

The last term is similar to the Coriolis force and is due to the zonal component u_1.

For the case $\mathbf{u} = \mathbf{v}$, we also give an alternative, coordinate-free expression of (2.1). Start with the following identity in $\mathbb{R}^3$

$$\mathbf{u}\cdot\nabla\mathbf{u} = \frac{1}{2}\nabla_{\mathbb{R}^3}|\mathbf{u}|^2 + (\nabla_{\mathbb{R}^3} \times \mathbf{u}) \times \mathbf{u}.$$

When project the above equation onto $T\mathbb{S}^2$, only the radially component of $(\nabla_{\mathbb{R}^3} \times \mathbf{u})$ matters, which is precisely $(\operatorname{curl}\mathbf{u})\,\mathbf{k}$. Thus, apply (1.2) to the previous equation and obtain,

$$\nabla_{\mathbf{u}}\mathbf{u} = \frac{1}{2}\nabla|\mathbf{u}|^2 + (\operatorname{curl}\mathbf{u})\,\mathbf{k} \times \mathbf{u}. \tag{2.2}$$

One of the benefits of this formulation is to easily obtain the vorticity equation from the (in)compressible Euler/Navier-Stokes equations on $\mathbb{S}^2$,

$$\partial_t\mathbf{u} + \nabla_{\mathbf{u}}\mathbf{u} + \nabla P = \nu\Delta\mathbf{u} + \mathbf{f}. \tag{2.3}$$

The hardest calculation is the nonlinear term $\operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u})$, which by (2.2) equals

$$\operatorname{curl}(\nabla_{\mathbf{u}}\mathbf{u}) = \operatorname{curl}((\operatorname{curl}\mathbf{u})\,\mathbf{k} \times \mathbf{u}) = \mathbf{u}\cdot\nabla(\operatorname{curl}\mathbf{u}) + (\operatorname{div}\mathbf{u})(\operatorname{curl}\mathbf{u})$$

where we also applied (1.17) and $\mathbf{k} \times \mathbf{u} = \mathbf{u}^{\perp}$. Finally, the equation for vorticity $\omega := \operatorname{curl}\mathbf{u}$ from (2.3) in $\mathbb{S}^2$ is

$$\partial_t\omega + \mathbf{u}\cdot\nabla\omega + (\operatorname{div}\mathbf{u})\,\omega = \nu\Delta\omega + \operatorname{curl}\mathbf{f},$$

as one would expect in $\mathbb{R}^2$.

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